Overview: row-space solution to $Ax = b$

This method begins with the realization that it's possible to effect linear operations on the rows of a matrix by left multiplying it with a suitable matrix. By extension, it's possible to perform a Gram-Schmidt orthonormalization of the rows of a matrix through such repeated operations.

In the case at hand, the system of linear equations is described by the matrix equation $Ax = b$, where A is the m-by-n coefficient matrix, x the n-dimensional vector of unknowns, and b is an m-dim vector. Also, take $m \leq n$, assume A has full row rank (the general case is treated in [1]), and assume all entries in A,b are real.

Determine a sequence of row operations, implemented by a sequence of m-by-m matrices M_k that orthonormalize the rows of A, and apply them from the left to the original equation to obtain

$$(\cdots M_2 M_1)Ax = (\cdots M_2 M_1)b \tag{1}$$

or $A'x = b'$, where $A' = MA$, $b' = Mb$, and $M = (\cdots M_2 M_1)$. (Keep in mind that the matrices M_s do not need to be *explicitly* created; only their actions on the rows need to be implemented.) At this point the reader should pause and realize that $x \in R^n$ and that there is an m-dimensional orthonormal basis available via the rows of A'. It follows that x may be expressed as

$$x = \alpha_1 v_1^T + \alpha_2 v_2^T + \cdots + \alpha_m v_m^T + \beta_1 w_1 + ... + \beta_{n-m} w_{n-m} \tag{2}$$

where the α_i and β_j are suitable coefficients, v_i are the rows of A', w_j represents a basis for the null space, $i = 1, \ldots, m$, and $j = 1, \ldots, n-m$. A simpler way to write out this solution is just to realize that the v_i^T are columns of $(A')^T$ and set

$$x = (A')^T \alpha + x_{null} \tag{3}$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)^T$, and x_{null} represents the previous sum $\sum \beta_j w_j$. (Another way to express x_{null} is Py, where P is the null space projection matrix $(1 - {A'}^T A')$ and y is an arbitrary n-dimensional vector.) After substituting this expression into $A'x = b'$, it follows that $\alpha = b'$, and the solution may be written as

$$x = (A')^T b' + x_{null} \tag{4}$$

Finally, observe that since the particular solution is built from the row space (and not the null space) of A, it is a *minimum norm* solution. This also follows from a characterization of its corresponding generalized inverse (see [1]).

References

[1] M.F. Zimmer, Two direct solvers for a system of linear equations. *arXiv preprint*, arXiv:1611.06633, 2020.

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