Examples for the *column space in situ* method

These notes supply examples for the *column space in situ* method for solving $Ax = b$, following the preprint by Zimmer[1]. The solutions will be based on this data:

$$A = \begin{bmatrix} 0 & -3i & 0 \\ 2i & 1 & -1 \\ 4i & 2-3i & -2 \end{bmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2i \\ 1+4i \end{pmatrix}$$

By inspection, $m = n = 3$. Also, A has rank 2, by design.

Main equations

The column space in situ method begins with the step of orthonormalizing the columns of A. The same column operations are performed on an identity matrix.

$$\begin{bmatrix} A \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} A' \\ M \end{bmatrix} \tag{1}$$

These column operations may be viewed as being effected by matrix multiplication by suitable matrices M_s. Thus, one may write $A' = AM$ and $M = (M_1 M_2 \cdots)$. As described in Zimmer [1], the solution may be written as

$$\begin{aligned} x_p &= M(A')^* \, b \\ x_h &= Py \\ P &= 1_n - GA \\ G &= M(A')^* \end{aligned}$$

where y is an arbitrary element of C^3. The particular solution may also be written as $x_p = Gb$.

The orthonormalization steps

This section explicitly shows the steps to orthonormalize A. It follows a *modified* Gram Schmidt (MGS) approach. They should be thought of as being applied to the augmented matrix in Eq. 1. Keep in mind that it is *not* necessary to explicitly construct the matrices M_i. They are shown here only for pedagogical purposes. The operations on each row c_j are:

$$\begin{aligned}
&\text{step 1: } c_1 \leftarrow c_1/\|c_1\| &&= c_1/(2\sqrt{5})\\
&\text{step 2: } c_2 \leftarrow c_2 - \langle c_2, c_1\rangle c_1 &&= c_2 + c_1(6/\sqrt{5} + i\sqrt{5})\\
&\text{step 3: } c_3 \leftarrow c_3 - \langle c_3, c_1\rangle c_1 &&= c_3 - c_1(i\sqrt{5})\\
&\text{step 4: } c_2 \leftarrow c_2/\|c_2\| &&= c_2(\frac{1}{3}\sqrt{\frac{5}{6}})
\end{aligned}$$

where $\langle v, w\rangle = w^* v$ and its computation only involves the portion of the column in A. It isn't necessary to perform any further steps that normally follow in a MGS method since at this point the columns are either part of an orthonormal set or they are 0. The corresponding matrices for these steps are:

$$M_1 = \begin{bmatrix} \frac{1}{2\sqrt{5}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & \frac{6}{\sqrt{5}} + i\sqrt{5} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 0 & -i\sqrt{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{\frac{5}{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The results

$$M = M_1 M_2 M_3 M_4 = \begin{bmatrix} \frac{1}{2\sqrt{5}} & \sqrt{\frac{5}{6}}\frac{(6+5i)}{30} & -\frac{i}{2} \\ 0 & \frac{1}{3}\sqrt{\frac{5}{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & -i\sqrt{\frac{5}{6}} & 0 \\ \frac{i}{\sqrt{5}} & \frac{2i}{5}\sqrt{\frac{5}{6}} & 0 \\ \frac{2i}{\sqrt{5}} & \frac{-i}{5}\sqrt{\frac{5}{6}} & 0 \end{bmatrix}$$

$$G = M(A')^* = \frac{1}{36}\begin{bmatrix} 6i-5 & 2-6i & -1-6i \\ 10i & -4i & 2i \\ 0 & 0 & 0 \end{bmatrix}$$

and the particular solution is

$$x_p = Gb = \begin{pmatrix} 5/6 \\ i/3 \\ 0 \end{pmatrix}$$

Also, the null space projection operator is

$$P = I_n - GA = \begin{bmatrix} 0 & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that Py, where y is an arbitrary elements of C^3, is proportional to the null space vector using the *row space* approach.

1 Online version

Again use the column-row expansion to expand the solution for x_p as

$$x_p = \sum_{j=1}^{n} x_p^{(j)} \tag{2}$$

$$x_p^{(j)} = \text{Col}_j[M] \ \text{Row}_j[(A')^*] \ b \tag{3}$$

Since the jth row of $(A')^*$ and jth column of M are available at the j-th step, $x_p^{(j)}$ can be computed at the j-th step. Also, assume b is available at the outset, and that the columns of A become available one at a time (from left to right).

Column 1 arrives

$$\text{step 1: } c_1 \leftarrow c_1/\|c_1\| \qquad = c_1/(2\sqrt{5})$$

Column 2 arrives

$$\text{step 2: } c_2 \leftarrow c_2 - \langle c_2, c_1\rangle c_1 = c_2 + c_1(6/\sqrt{5} + i\sqrt{5})$$

$$\text{step 3: } c_2 \leftarrow c_2/\|c_2\| \qquad = c_2(\frac{1}{3}\sqrt{\frac{5}{6}})$$

Column 3 arrives

$$\text{step 4: } c_3 \leftarrow c_3 - \langle c_3, c_1\rangle c_1 = c_3 - c_1(i\sqrt{5})$$

Finally note that in computing x_p it is more efficient to first multiply b against $\text{Row}_j[(A')^*]$ in equation (3), and then multiply the result against $\text{Col}_j[M]$. Also observe that this same online technique can be trivially extended to compute G, P or even x_h in an online manner.

(Double-hyphens in a matrix mean that no data has been entered there yet.) Also, the "intermediate results" are how A and x_p appear following the j-th update. After $j = 3$, the updates are complete, and the last A may be identified with A'.

$j = 1$ ———————————————

input data:

$$\text{Col}_1(A) = \begin{bmatrix} 0 \\ 2i \\ 4i \end{bmatrix}$$

intermediate results:

$$A' = \begin{bmatrix} 0 & -- & -- \\ \frac{i}{\sqrt{5}} & -- & -- \\ \frac{2i}{\sqrt{5}} & -- & -- \end{bmatrix}, \quad M = \begin{bmatrix} \frac{1}{2\sqrt{5}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$x_p^{(1)} = \text{Col}_1[M]\ \text{Row}_1[(A')^*]\ b = \begin{pmatrix} 1 - \frac{i}{5} \\ 0 \\ 0 \end{pmatrix}$$

$j = 2$ ———————————————

input data:

$$\text{Col}_2(A) = \begin{bmatrix} -3i \\ 1 \\ 2 - 3i \end{bmatrix}$$

intermediate results:

$$A' = \begin{bmatrix} 0 & -i\sqrt{\frac{5}{6}} & -- \\ \frac{i}{\sqrt{5}} & \frac{2i}{5}\sqrt{\frac{5}{6}} & -- \\ \frac{2i}{\sqrt{5}} & -\frac{i}{5}\sqrt{\frac{5}{6}} & -- \end{bmatrix}, \quad M = \begin{bmatrix} \frac{1}{2\sqrt{5}} & \frac{6+5i}{30}\sqrt{\frac{5}{6}} & 0 \\ 0 & \frac{1}{3}\sqrt{\frac{5}{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$x_p^{(2)} = \text{Col}_2[M]\ \text{Row}_2[(A')^*]\ b = \begin{pmatrix} \frac{i}{5} - \frac{1}{6} \\ \frac{i}{3} \\ 0 \end{pmatrix}$$

$j = 3$ ———————————————
input data:

$$\text{Col}_3(A) = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}$$

intermediate results:

$$A' = \begin{bmatrix} 0 & -\sqrt{\frac{6}{5}} & 0 \\ \frac{i}{\sqrt{5}} & \frac{2i}{5}\sqrt{\frac{6}{5}} & 0 \\ \frac{2i}{\sqrt{5}} & -\frac{i}{5}\sqrt{\frac{6}{5}} & 0 \end{bmatrix}, \quad M = \begin{bmatrix} \frac{1}{2\sqrt{5}} & \frac{6+5i}{30}\sqrt{\frac{5}{6}} & -\frac{i}{2} \\ 0 & \frac{1}{3}\sqrt{\frac{5}{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$x_p^{(3)} = \text{Col}_3[M]\ \text{Row}_3[(A')^*]\ b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The final result for the particular solution is

$$x = x_p^{(1)} + x_p^{(2)} + x_p^{(3)} = \begin{pmatrix} 5/6 \\ i/3 \\ 0 \end{pmatrix}$$

which is the same as found earlier. In addition, G can be computed online, following [1]; it yields the same result.

Exercises

(1) Verify $Ax_p = b$ and $Ax_h = 0$ for all approaches
(2) Use $A'(A')^*$ to show that b lies in the column space of A
(3) Verify that G is a generalized inverse of type $\{123\}$. What are the properties of the solution Gb formed from it? Repeat this for the row space solution.
(4) Explain the differences between the particular solution found here and that found with the *row space* method.

References

[1] M.F. Zimmer, Two direct solvers for a system of linear equations. *arXiv preprint*, arXiv:1611.06633, 2020.

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